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A Recursive Approach to the Solution of Abstract Linear Equations and the Tau Method

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Abstract—Earlier work of Ortiz [1,2] and his collaborators [3,4] is generalized and extended for a recursive approach to the solution of abstract linear equations. Two families of vectors are simultaneously generated by means of Noether bases in the context of well-ordered bases. They are the families of Ortiz canonical vectors and residual vectors associated with every linear mapping on infinite-dimensional vector spaces. The latter would serve to determine a necessary and sufficient condition which ensures the existence of solutions and the former to provide a direct representation of a solution. Such a representation of a preimage vector admits the same scalar-cooredinates and the same index as its correposnding image vector, omitting the scalars of nonaccessible oindices. It is shown that every linear mapping is uniquely associated with a pair whose components are a family of cosets of Ortiz canonical vectors and a family of residual vectors, for any given well-ordered basis of its codomain space. the terms of the above-mentioned families are reproduced by self-starting recursive relations in the context of standard bases. Our abstract results show that the recursive relations between the elements of the families mentioned above are conveniently generated through *matrices in row echelon form.* The former makes possible the recursive construction of the solution for an extensive class of linear operator equations, including equations determined by operators of infinite kernel index and deficiency. Several examples from different fields of applications, such as *algebraic systems and partial differential equations with bivariate polynomial coefficients*, are used to illustrate our method.

Keywords—Recursive approach, Infinite-dimensional vector spaces, Abstract linear equations, Operator linear equations, Noether basis, Ortiz canonical basis, Tau method, Infinite matrices in echelon form.

1. INTRODUCTION

Canonical polynomials were introduced heuristically by C. Lanczos in the early 1940s and discussed in his book [5] of 1956 in connection with an ingenious technique, called the *Tau method*, he pioneered and which since then has been extensively used in scientific computation. These polynomials have the ability to express the Tau method's approximate solutions of differential problems in a very direct way.

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A general definition of the canonical polynomials associated with differential operators $D = \sum_{i=0}^{n} p_i(x)\frac{d^i}{dx^i}$ with polynomial coefficients was given by Ortiz in [2]. He showed there that such a class of operators is in one-to-one correspondence with a class of sequences whose terms are classes of equivalence of canonical polynomials having the algebraic Kernel of an operator as their modulo. Ortiz also demonstrates, in the reference cited above, that the individual elements of such a sequence can be generated by a simple self-starting recursive procedure, which makes them attractive from a computational point of view.

In [3], Llorente and Ortiz discussed the canonical polynomials in connection with injective endomorphisms, E, of the space of polynomials of finite deficiency. They showed that the sequence of canonical polynomials is a basis of the space of polynomials uniquely determined by E (see also [6]). Ortiz and Samara formulated an operational approach to the Tau method in [4] and introduced a special type of *infinite banded matrices*, showing that such matrices represent uniquely differential operators of the type mentioned above. They have also used the operational approach to the Tau method [7] for the numerical solution of partial differential equations with variable coefficients.

In this paper, we shall be concerned with a purely algebraic extension of Ortiz' theory on the recursive generation of the canonical polynomials to the general case of linear mappings defined on abstract vector spaces. We consider two abstract vector spaces X, Y over the same field of scalars and an arbitrary linear mapping T of X into Y. Assuming the axiom of choice, Y admits a Hamel basis $\mathbf{b} = (\mathbf{b}_i)_{i\in I}$ over a well-ordered indexing set I. The well-ordering of I, ensured by Zermelo's *well-ordering* theorem, would serve to define the subset S of I, called the *set of nonaccessible indices*. The latter generates an algebraic complement of the range of T by means of the elements $(\mathbf{b}_s)_{s\in S}$ and is denoted by $\mathcal{R}_S$. The space $\mathcal{R}_S$, called *residual space*, enables us to define a Noether basis $\mathbf{v} = (\mathbf{v}_i)_{i\in I\setminus S}$ of the range of T, that in turn generates implicitly a family $\mathbf{q} = (\mathbf{q}_i)_{i\in I\setminus S}$ of Ortiz canonical vectors in X simultaneously with a family $\mathbf{r} = (\mathbf{r}_i)_{i\in I\setminus S}$ of residual vectors in $\mathcal{R}_S$.

It is shown that the above-introduced families of vectors yield the following remarkable properties.

(1) Given a basis $\mathbf{b} = (\mathbf{b}_i)_{i\in I}$ of Y, every linear mapping T of X to Y is uniquely associated with the pair of families $(\mathcal{L}, \mathbf{r})$, where $\mathcal{L}$ is a family of cosets of Ortiz canonical vectors modulo Kernel of T (Theorem 3).

(2) A family of Ortiz canonical vectors is extendable to a basis $\tilde{\mathbf{q}}$ of X by means of elements of a basis of the null space of T (Theorem 4).

(3) The families $\mathcal{L}$, $\mathbf{r}$ are also generated through self-starting recursive relations (Theorem 6) associated with abstract linear mappings. These relations are formulated with the aid of a standard basis $\tilde{\mathbf{e}}$ of X.

(4) A necessary and sufficient condition for the existence of solutions to an abstract linear equation

$$T(\mathbf{x}) = \mathbf{y} \tag{1}$$

is formulated with the aid of the family of residual vectors. It enables us to determine the parametric form of the scalar-coordinates of the right-hand side vectors $\mathbf{y}$, for which exact solutions exist. Moreover, for every $\mathbf{y} = \sum_{i\in I}\alpha_i\mathbf{b}_i$ in the range of T, a solution to (1), expressed in terms of $\mathbf{q}$, yields the same coordinates and of the same index,[1] as the vector $\mathbf{y}$ except of those whose index ranges over S (Theorem 7), namely,

$$\mathbf{x} = \sum_{i\in I\setminus S}\alpha_i\mathbf{q}_i$$

plus an arbitrary linear combination of elements of the null space of T.

[1] This property justifies the choice of the terminology 'Ortiz canonical vectors'.

Our results are also formulated in connection with *matrices in row echelon form.* A general definition of infinite matrices in row echelon form over well-ordered indexing sets is followed by a useful criterion for the detection of such matrices. It is shown that the matrix representation of an abstract linear mapping, relative to the bases $(\tilde{\mathbf{e}}, \mathbf{b})$, is in prerow echelon form (Theorem 9). The latter is an existence theorem of matrices in row echelon form associated with abstract linear mappings. Moreover, the intimate relation between standard bases and matrices in row echelon form leads to a constructive scheme for the recursive formulation of the solution of operator equations. Banded infinite matrices, which represent ordinary differential operators with polynomial coefficients and blocks of such matrices, which represent matrix differential operators with entries operators of the above-mentioned type, can be transformed into matrices in row echelon form by means of a finite and constructive procedure.

The formulation of Ortiz' representation theorems of the Tau method in the context of abstract linear equations sets an extensive framework for the numerical treatment of equations involving linear and nonlinear operators. Our study was initially motivated by the problems in the extension of Ortiz recursive formulation of the Tau method to systems of ordinary differential equations (see [8] and the references given there). Such extension is derived in the foregoing reference within the framework of the general ideas presented in this paper. As a consequence of our work, the restrictive condition of the finiteness of the deficiency of a linear operator, namely, $\mathrm{card}(S) < \infty$, which characterizes ordinary differential operators with polynomial coefficients, is not a prerequisite for the construction of the recursive formulae of the Ortiz canonical and residual polynomials. The latter makes possible the extension of the Tau technique [7,9] to a wider class of operator equations, including the case of partial differential equations of infinite deficiency (see Example 4).

In a purely algebraic direction, our results would serve to show in [10] existence and uniqueness theorems of infinite matrices in reduced row echelon form infinite matrices associated with abstract linear mappings.

2. ORTIZ CANONICAL AND RESIDUAL VECTORS: EXISTENCE, UNIQUENESS, RECURRENCE

A sharp partial order relation $\prec$ on a set I is a binary relation on I whose elements[2] satisfy the following two conditions:

(i) if $(i,j) \in \prec$, then $(j,i) \notin \prec$ (asymmetry), and
(ii) if $(i,j) \in \prec$ and $(j,k) \in \prec$, then $(i,k) \in \prec$ (transitivity).

A blunt order relation $\preceq$ on I associated with a sharp order $\prec$ is an extension of $\prec$ by means of all pairs (i,i), $i \in I$, i.e., $\preceq\, = \,\prec \cup\, id_I$, where $id_I = \{(i,i),\ i \in I\}$. Thus, the notation $i \prec j$ is equivalent to $i \preceq j$ and $i \neq j$.

A partially ordered set $(I, \prec)$ is called *well ordered* if and only if for each $J \subset I$ with $J \neq \emptyset$ there exists a $j_0 \in J$ such that $j_0 \preceq j$ for every $j \in J$; that is, each nonempty subset of I has a first element. A consequence of the above definition is that an element of a well-ordered set I that has a successor in I has an immediate successor, but it need not have an immediate predecessor. An initial interval $I(i)$ of a well-ordered set I determined by i is the set of all predecessors of i, which is formally defined by $I(i) = \{j \in I : j \prec i\}$. An equivalent version of the set-theoretic axiom of choice is Zermelo's well-ordering theorem, which shows that there is a relation which well-orders any set.

In a theory $\mathcal{T}$ in which a set I is well ordered, propositions are frequently shown with the aid of the *transfinite induction* principle [11, pp. 148–157], which is usually stated in the following form. Let $I(i)$ be an initial interval of I determined by i. Let also $\{P(i),\ i \in I\}$ be a set of propositions and i_0 be the first element of I. Let us assume both

[2]The notation $i \prec j$ is equivalently used with the set-theoretic notation $(i,j) \in \prec$.

(i) $P(i_0)$ is true, and
(ii) for an arbitrary i, the hypothesis $P(j)$, $j \in I(i)$ is true implies $P(i)$ is also true.

Then $P(i)$ holds true for all $i \in I$.

2.1. The Generation of Ortiz Canonical and Residual Vectors Associated with Linear Mappings

In this paper, an ordered basis (or simply basis) $\mathbf{b} = (\mathbf{b}_i)_{i\in I}$ of a vector space Y is defined to be a linearly independent and generating family of vectors of Y whose indexing set is well ordered and which is denoted by $(I, \prec)$. Thus, each order relation on the indexing set I of $\mathbf{b}$ generates a distinct ordered basis of Y. In the same fashion we shall treat families of vectors over well-ordered indexing sets. Let $\mathbf{b} = (\mathbf{b}_i)_{i\in I}$ be a basis of a vector space Y. An element $\mathbf{y}$ of Y can be expressed by $\sum_{i\in I} \alpha_i \mathbf{b}_i$, assuming that a finite number of scalars α_i are nonzero. The set $\operatorname{supp}(\mathbf{y}) = \{i \in I : \alpha_i \neq 0\}$ is called *finite support* of $\mathbf{y}$. If $\mathbf{y}$ is a nonzero vector, then $\operatorname{supp}(\mathbf{y})$ is a nonempty and finite subset of I. Consequently, $\operatorname{supp}(\mathbf{y})$ yields a unique greatest element, relative to $(I, \prec)$, which will be denoted by $\operatorname{maxsupp}(\mathbf{y})$.

DEFINITION 1. *Let G be a subspace of Y. The subset S of I defined by $S = \{s \in I : s \neq \operatorname{maxsupp}(\mathbf{g}), \forall\, \mathbf{g} \in G\}$ is called the set of nonaccessible indices of G, relative to $\mathbf{b}$. Equivalently, the set S is characterized as follows: $s \in S$ if and only if there are not elements in G of the form $\mathbf{b}_s + \sum_{j\in I(s)} \alpha_j \mathbf{b}_j$, where $I(s)$ is an initial interval of I determined by s.*

REMARK 1. In general, two distinct well-orderings of the indexing set I of a basis $\mathbf{b}$ of Y generate different sets of nonaccessible indices of a subspace G of Y, as the index $\operatorname{maxsupp}(\mathbf{g})$ may vary while $\mathbf{g} \in G$ remains constant (see Example 1).

A subset J of a well-ordered set $(I, \prec)$ is well ordered with reference to the order relation induced on J by $\prec$ and which will be denoted by $(J, \prec_J)$. The space spanned by $(\mathbf{b}_s)_{s\in S}$ is denoted by $\mathcal{R}_S$ and is called *residual space* of G, relative to $\mathbf{b}$. For each $i \in I \setminus S$, the definition of S implies the existence of a vector, say $\boldsymbol{\varepsilon}_i$, in G such that $\operatorname{maxsupp}(\boldsymbol{\varepsilon}_i) = i$ and we write for it $\boldsymbol{\varepsilon}_i = \sum_{k\preceq i} \alpha_{ik} \mathbf{b}_k$, with $\alpha_{ii} \neq 0$. The mapping $\boldsymbol{\varepsilon} : I \setminus S \ni i \mapsto \boldsymbol{\varepsilon}_i \in G$ defines a family of vectors $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_i)_{i\in I\setminus S}$ whose indexing set is well ordered by $\prec_{I\setminus S}$.

LEMMA 1.

(i) *The space $\mathcal{R}_S$ is an algebraic complement of G, namely,*

$$Y = G \oplus \mathcal{R}_S.$$

(ii) *The family $(\boldsymbol{\varepsilon}_i)_{i\in I\setminus S}$ is a basis of G.*

PROOF.

(i). We shall show that the family $(\boldsymbol{\varepsilon}_i)_{i\in I\setminus S} \cup (\mathbf{b}_s)_{s\in S}$ is a basis of Y. After the identification of the elements $\mathbf{b}_s$ with $\boldsymbol{\varepsilon}_s$ for all $s \in S$, the above family can be denoted by $(\boldsymbol{\varepsilon}_i)_{i\in I}$. Clearly, for every $i \in I$ it follows that $\operatorname{maxsupp}(\boldsymbol{\varepsilon}_i) = i$. It suffices to argue by transfinite induction on $i \in I$.

THE LINEAR INDEPENDENCE OF $\boldsymbol{\varepsilon}$. Let i_0 be the first element of I. It follows that $\boldsymbol{\varepsilon}_{i_0} = \lambda_{i_0} \mathbf{b}_{i_0}$ for $\lambda_{i_0} \neq 0$ ($\lambda_{i_0} = 1$, if $i_0 \in S$, or $\lambda_{i_0} = \alpha_{i_0 i_0}$, if $i_0 \in I \setminus S$) and the assertion follows for $i = i_0$. The inductive hypothesis states that $(\boldsymbol{\varepsilon}_k)_{k\in I(i)}$ is a linearly independent family. We assume that $\sum_{k\preceq i} \mu_k \boldsymbol{\varepsilon}_k = 0$, which is equivalent to

$$\mu_i \boldsymbol{\varepsilon}_i = -\sum_{k\in I(i)} \mu_k \boldsymbol{\varepsilon}_k. \tag{2}$$

Taking into account that $\boldsymbol{\varepsilon}_i = \alpha_{ii}\mathbf{b}_i + \sum_{k\in I(i)} \alpha_{ik} \mathbf{b}_k$, after suitable algebraic manipulations, equation (2) takes the form

$$\mu_i \mathbf{b}_i = -\mu_i \alpha_{ii}^{-1} \sum_{k\in I(i)} \alpha_{ik} \mathbf{b}_k - \alpha_{ii}^{-1} \sum_{k\in I(i)} \mu_k \boldsymbol{\varepsilon}_k. \tag{3}$$

The right-hand side of equation (3) is an element of $\mathrm{span}\{(\mathbf{b}_k)_{k\in I(i)}\}$, and therefore, $\mu_i\mathbf{b}_i \in \mathrm{span}\{(\mathbf{b}_k)_{k\in I(i)}\}$. Now, μ_i must be zero; otherwise, $\mathbf{b}_i \in \mathrm{span}\{(\mathbf{b}_k)_{k\in I(i)}\}$, a fact contradictory to the linear independence of $(\mathbf{b}_i)_{i\in I}$. Thus, (2) should be of the form $\sum_{k\in I(i)} \mu_k\boldsymbol{\varepsilon}_k = 0$. Finally, the inductive hypothesis implies that $\mu_k = 0$ for all $k \in I(i)$ and the induction is complete.

The family $\boldsymbol{\varepsilon}$ is a generating system of Y. It suffices to show that every $\mathbf{b}_i$ can be expressed as a linear combination of $\boldsymbol{\varepsilon}_j$. As $\boldsymbol{\varepsilon}_{i_0} = \lambda_{i_0}\mathbf{b}_{i_0}$ for some $\lambda_{i_0} \neq 0$, the assertion follows for $i = i_0$. The inductive hypothesis states that for each $k \in I(i)$, $\mathbf{b}_k$ can be expressed as a linear combination of $\boldsymbol{\varepsilon}_j$. The expression $\boldsymbol{\varepsilon}_i = \sum_{k\preceq i} \alpha_{ik}\mathbf{b}_k$ can be written as

$$\mathbf{b}_i = -\alpha_{ii}^{-1} \sum_{k\in I(i)} \alpha_{ik}\mathbf{b}_k + \alpha_{ii}^{-1}\boldsymbol{\varepsilon}_i. \tag{4}$$

The inductive hypothesis shows that the right-hand side of (4) is expressible as a linear combination of $\boldsymbol{\varepsilon}_j$. Thus, the family $\boldsymbol{\varepsilon}$ is a generating system of Y, and therefore, a basis of Y. Now, each $\mathbf{y} \in Y$ can be expressed uniquely as $\mathbf{y} = \sum_{i\in I} \lambda_i\boldsymbol{\varepsilon}_i = \sum_{i\in I\setminus S} \lambda_i\boldsymbol{\varepsilon}_i + \sum_{s\in S} \lambda_s\mathbf{b}_s$. Taking $\mathbf{g} = \sum_{i\in I\setminus S} \lambda_i\boldsymbol{\varepsilon}_i$ and $\mathbf{r} = \sum_{s\in S} \lambda_s\mathbf{b}_s$, since $\mathbf{g} \in G$ and $\mathbf{r} \in \mathcal{R}_S$, it follows that $\mathbf{y}$ is uniquely expressible in the form $\mathbf{y} = \mathbf{g} + \mathbf{r}$. Thus, $Y = G \oplus \mathcal{R}_S$.

(ii). The subfamily $(\boldsymbol{\varepsilon}_i)_{i\in I\setminus S}$ of $\boldsymbol{\varepsilon}$ is obviously linearly independent. Let $\mathbf{y} \in G$. Then it is expressible as $\mathbf{y} = \sum_{i\in I} \lambda_i\boldsymbol{\varepsilon}_i = \sum_{i\in I\setminus S} \lambda_i\boldsymbol{\varepsilon}_i + \sum_{s\in S} \lambda_s\mathbf{b}_s$ and so $\mathbf{y} - \sum_{i\in I\setminus S} \lambda_i\boldsymbol{\varepsilon}_i \in \mathcal{R}_S$. Since $\mathbf{y}$ and $\boldsymbol{\varepsilon}_i$ for $i \in I \setminus S$ are elements of G, it follows that $\mathbf{y} - \sum_{i\in I\setminus S} \lambda_i\boldsymbol{\varepsilon}_i = 0$. Thus, $(\boldsymbol{\varepsilon}_i)_{i\in I\setminus S}$ is a generating system of G, and therefore, a basis of G. ∎

REMARK 2. In the classical proof, the existence of an algebraic complement of a subspace G of Y, spanned by a part of a given basis $\mathbf{b}$ of Y, is deduced from Zorn's lemma (see [12, pp. 238–242]. However, it is not specified in it which part of $\mathbf{b}$ spans an algebraic complement of G. Alternatively, Zermelo's well-ordering theorem enabled us to define the set S of nonaccessible indices, which in turn generates the algebraic complement $\mathcal{R}_S$ of G. Furthermore, as we shall see in Section 3, the set S is constructible in the context of infinite matrices in row echelon form (see also the examples of Section 4).

Let us consider the subfamily $(\mathbf{b}_i)_{i\in I\setminus S}$ of $\mathbf{b}$. By virtue of Lemma 1, if $i \in I \setminus S$, then $\mathbf{b}_i$ can take a unique form

$$\mathbf{b}_i = \mathbf{v}_i - \mathbf{r}_i \tag{5}$$

for some $\mathbf{v}_i \in G$ with $\mathbf{v}_i \neq 0$ and $\mathbf{r}_i \in \mathcal{R}_S$; if $i \in S$, then $\mathbf{b}_i \in \mathcal{R}_S$. Additionally, the following remarkable result is due to Noether [12, Theorem 1, p. 242].

THEOREM 1. NOETHER. *The family* $\mathbf{v} = (\mathbf{v}_i)_{i\in I\setminus S}$ *defined by (5) is a basis of* G, *called Noether basis, relative to* $\mathbf{b}$.

Throughout this paper, we assume that X and Y are abstract vector spaces. The class of linear mappings of X to Y over the same field of scalars F is called the class of homomorphisms and is denoted by $\mathrm{Hom}_F(X,Y)$. The range and the null spaces of $T \in \mathrm{Hom}_F(X,Y)$ are denoted by $\mathrm{im}(T)$ and $\mathrm{Ker}(T)$, respectively. Let S be the set of nonaccessible indices of the subspace $\mathrm{im}(T)$ of Y, relative to $\mathbf{b}$. The algebraic complement $\mathcal{R}_S$ of $\mathrm{im}(T)$, defined above, is called *residual space* of T, relative to $\mathbf{b}$. Accordingly, the cardinality of S indicates the *deficiency* of T. A vector $\mathbf{r}_i \in \mathcal{R}_S$ for $i \in I\setminus S$, defined by (5), is called *residual vector of index* i associated with T, relative to $\mathbf{b}$. Let $X/\mathrm{Ker}(T)$ be the quotient space of X modulo $\mathrm{Ker}(T)$. The canonical isomorphism $\tilde{T}$ induced by T is defined by $\tilde{T} : X/\mathrm{Ker}(T) \mapsto \mathrm{im}(T)$, $\tilde{T}([\mathbf{x}]) = T(\mathbf{x})$, where $[\mathbf{x}] \in X/\mathrm{Ker}(T)$. Let $\mathbf{v} = (\mathbf{v}_i)_{i\in I\setminus S}$ be a Noether basis of $\mathrm{im}(T)$, relative to $\mathbf{b}$. Following Ortiz [2], the classes of equivalence $\mathcal{L}_i = \tilde{T}^{-1}(\mathbf{v}_i)$ for $i \in I \setminus S$ modulo $\mathrm{Ker}(T)$ will be called *Lanczos' cosets associated with* T, *relative to* $\mathbf{b}$.

By virtue of the axiom of choice, from each Lanczos' coset $\mathcal{L}_i$ for $i \in I \setminus S$, we choose a vector $\mathbf{q}_i$. The foregoing procedure gives rise to a mapping $\mathbf{q} : I \setminus S \ni i \mapsto \mathbf{q}_i \in X$, which, in turn, defines an indexed family $\mathbf{q} = (\mathbf{q}_i)_{i\in I\setminus S}$ whose indexing set is well ordered by $\prec_{I\setminus S}$. The former is called family of *Ortiz canonical vectors associated with* T, *relative to* $\mathbf{b}$. Accordingly, each Lanczos' coset $\mathcal{L}_i$, $i \in I \setminus S$, consists of Ortiz canonical vectors of the same index and we write formally for it $\mathcal{L}_i = \mathbf{q}_i + \mathrm{Ker}(T)$. Moreover, the elements of $\mathbf{q}$ satisfy the equation

$$T(\mathbf{q}_i) = \mathbf{v}_i, \tag{6}$$

for all $i \in I \setminus S$. In view of (5), an Ortiz canonical vector $\mathbf{q}_i$ can be equivalently defined by the equation[3]

$$T(\mathbf{q}_i) = \mathbf{b}_i + \mathbf{r}_i, \tag{7}$$

for $i \in I \setminus S$ and $\mathbf{r}_i \in \mathcal{R}_S$.

Henceforth, Noether's Theorem 1 guarantees the existence of Ortiz canonical vectors associated with an abstract linear mapping, as being preimages of a Noether basis of its range.

2.2. Properties of Ortiz Canonical Vectors

Following Remark 1, let S_1 and S_2 be two sets of nonaccessible indices generated by two ordered bases of Y, say $(\mathbf{b}_i)_{i\in I_1}$, $(\mathbf{b}_i)_{i\in I_2}$, where I_1, I_2 indicate the same indexing set I equipped with two distinct well-orders $\prec_1$ and $\prec_2$, respectively. There are cases in which different well-orderings of I determine the same set S (see Examples 2 and 3 below). A relevant result is demonstrated in the following theorem.

THEOREM 2. *Let* $\mathbf{q}^{(1)} = (\mathbf{q}_i^{(1)})_{i\in I\setminus S_1}$, $\mathbf{q}^{(2)} = (\mathbf{q}_i^{(2)})_{i\in I\setminus S_2}$ *be two families of Ortiz canonical vectors generated by* $(\mathbf{b}_i)_{i\in I_1}$, $(\mathbf{b}_i)_{i\in I_2}$, *respectively. If* $S_1 = S_2$, *then two Ortiz canonical vectors chosen from* $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$ *have the same index if and only if they differ by an element of* $\mathrm{Ker}(T)$.

PROOF. Let us choose two Ortiz canonical vectors of the same index, say $\mathbf{q}_i^{(1)} \in \mathbf{q}^{(1)}$, $\mathbf{q}_i^{(2)} \in \mathbf{q}^{(2)}$. As $S = S_1 = S_2$, it follows that S_1, S_2 generate the same residual space $\mathcal{R}_S$. Thus, for each $i \in I \setminus S$, $\mathbf{b}_i$ is uniquely expressible in terms of elements of $\mathrm{im}(T)$ and $\mathcal{R}_S$. It follows from (5) and (6) that $T(\mathbf{q}_i^{(1)}) = T(\mathbf{q}_i^{(2)}) = \mathbf{v}_i$. Thus, $T(\mathbf{q}_i^{(1)}) = T(\mathbf{q}_i^{(2)}) \implies T(\mathbf{q}_i^{(1)} - \mathbf{q}_i^{(2)}) = 0 \implies \mathbf{q}_i^{(1)} - \mathbf{q}_i^{(2)} \in \mathrm{Ker}(T)$. Conversely, if two Ortiz canonical vectors differ by an element of $\mathrm{Ker}(T)$, they belong to the same Lanczos' coset, and therefore, they yield the same index. ∎

Let us denote by $\mathcal{L} = (\mathcal{L}_i)_{i\in I\setminus S}$ a family of Lanczos' cosets associated with T, relative to $\mathbf{b}$.

THEOREM 3. UNIQUENESS. *Given a basis* $\mathbf{b} = (\mathbf{b}_i)_{i\in I}$ *of* Y, *there exists a unique pair* $(\mathcal{L}, \mathbf{r})$ *associated with* T, *relative to* $\mathbf{b}$.

PROOF. Let us consider $T \in \mathrm{Hom}_F(X,Y)$ associated with the pairs $\mathcal{L}^{(1)} = (\mathcal{L}_i^{(1)})_{i\in I\setminus S_1}$, $\mathbf{r}^{(1)} = (\mathbf{r}_i^{(1)})_{i\in I\setminus S_1}$, and $\mathcal{L}^{(2)} = (\mathcal{L}_i^{(2)})_{i\in I\setminus S_2}$, $\mathbf{r}^{(2)} = (\mathbf{r}_i^{(2)})_{i\in I\setminus S_2}$, respectively. Given a basis $\mathbf{b}$ of Y, its indexing set I is equipped with a fixed well order, and therefore, the set S of nonaccessible indices is uniquely associated with T, relative to $\mathbf{b}$ (Definition 1). Consequently, $S_1 = S_2 = S$. Let $\tilde{T}$ be the canonical isomorphism induced by T. As $\tilde{T}(\mathcal{L}_i) = T(\mathbf{q}_i)$, $i \in I \setminus S$, the relation (7) implies $\mathbf{b}_i = \tilde{T}(\mathcal{L}_i^{(1)}) - \mathbf{r}_i^{(1)}$ and $\mathbf{b}_i = \tilde{T}(\mathcal{L}_i^{(2)}) - \mathbf{r}_i^{(2)}$ for all $i \in I \setminus S$. By virtue of Lemma 1, it follows that $\mathbf{b}_i$ is uniquely expressible in terms of elements of $\mathrm{im}(T)$ and $\mathcal{R}_S$, and thus, $\tilde{T}(\mathcal{L}_i^{(1)}) = \tilde{T}(\mathcal{L}_i^{(2)})$ and $\mathbf{r}_i^{(1)} = \mathbf{r}_i^{(2)}$ for all $i \in I \setminus S$. As $\tilde{T}$ is an isomorphism, we also have $\mathcal{L}_i^{(1)} = \mathcal{L}_i^{(2)}$ for all $i \in I \setminus S$ and the equalities $\mathcal{L}^{(1)} = \mathcal{L}^{(2)}$ and $\mathbf{r}^{(1)} = \mathbf{r}^{(2)}$ follow. ∎

Let $\mathbf{u} = (\mathbf{u}_w)_{w\in W}$ be a basis of $\mathrm{Ker}(T)$, where W is equipped with a well order, say $\prec_W$. We define the extended family $\tilde{\mathbf{q}} = \mathbf{u} \cup \mathbf{q}$. The indexing set of $\tilde{\mathbf{q}}$ is the disjoint union of W, $I \setminus S$, denoted by K. An order relation on K is defined by the binary relation $<_K = \prec_W \cup \prec_{I\setminus S} \cup (W \times (I \setminus S))$. The set K is formally well-ordered by $<_K$, in such a way that each element of W is a predecessor of every element of $I \setminus S$.

[3]This equation generalizes (6) in Definition 3.1 of the canonical polynomials, introduced by Ortiz in [2].

THEOREM 4.

(i) *The family* $\mathcal{L}$ *is a basis of* $X/\operatorname{Ker}(T)$.

(ii) *A family* $\mathbf{q}$ *of Ortiz canonical vectors is linearly independent and generates an algebraic complement of* $\operatorname{Ker}(T)$, *namely,*

$$X = \operatorname{span}\{\mathbf{q}\} \oplus \operatorname{Ker}(T). \tag{8}$$

(iii) *The extended family* $\tilde{\mathbf{q}}$ *is a basis of* X.

PROOF.

(i) Let $\mathbf{v}$ be a Noether basis of $\operatorname{im}(T)$. As $\tilde{T}$ is an isomorphism and $\mathcal{L} = \tilde{T}^{-1}(\mathbf{v})$, the assertion follows.

(ii) Let $\sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i \in \operatorname{Ker}(T)$ for some $\alpha_i \in F$. The foregoing hypothesis is equivalent to

$$\sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i + \operatorname{Ker}(T) = \operatorname{Ker}(T). \tag{9}$$

Taking into account that $\operatorname{Ker}(T) = [0]$ (the zero element of $X/\operatorname{Ker}(T)$), with the aid of $\sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i + \operatorname{Ker}(T) = \sum_{i\in I\setminus S} \alpha_i(\mathbf{q}_i + \operatorname{Ker}(T)) = \sum_{i\in I\setminus S} \alpha_i \mathcal{L}_i$, relation (9) can be rewritten as $\sum_{i\in I\setminus S} \alpha_i \mathcal{L}_i = [0]$. Since $(\mathcal{L}_i)_{i\in I\setminus S}$ is a linearly independent family, it follows that $\alpha_i = 0$ for all $i \in I \setminus S$. Thus,

$$\operatorname{span}\{\mathbf{q}\} \cap \operatorname{Ker}(T) = \{0\}. \tag{10}$$

Let us consider an arbitrary element $\mathbf{x}$ of X. As $(\mathcal{L}_i)_{i\in I\setminus S}$ is a generating family of $X/\operatorname{Ker}(T)$, we have $\mathbf{x} + \operatorname{Ker}(T) = \sum_{i\in I\setminus S} \alpha_i \mathcal{L}_i = \sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i + \operatorname{Ker}(T)$. Thus, $\mathbf{x} - \sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i \in \operatorname{Ker}(T)$. The latter implies $\mathbf{x} = \sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i + \sum_{w\in W} c_w \mathbf{u}_w$, where $(\mathbf{u}_w)_{w\in W}$ is a basis of $\operatorname{Ker}(T)$. Accordingly,

$$X = \operatorname{span}\{\mathbf{q}\} + \operatorname{Ker}(T), \tag{11}$$

and therefore, (8) follows from (10) and (11). To show the linear independence of $\mathbf{q}$, let us consider $\sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i = 0$. It follows that $\sum_{i\in I\setminus S} \alpha_i \mathbf{q}_i \in \operatorname{Ker}(T)$. As before, relation (9) implies $\alpha_i = 0$ for all $i \in I \setminus S$ and the assertion follows.

(iii) Since $\mathbf{q}$ and $\mathbf{u}$ are bases of factor subspaces of X [13, Proposition 19, p. 217], it follows that their union $\tilde{\mathbf{q}}$ is a basis of X. ∎

The above-generated ordered basis $\tilde{\mathbf{q}}$ of X will be called *Ortiz canonical basis* associated with T, relative to $\mathbf{b}$.

REMARK 3. It should be noted that a Noether basis of $\operatorname{im}(T)$ is not necessarily required for the proof of Theorem 4. Therefore, the above theorem holds for any choice of a basis $\mathbf{v}$ of $\operatorname{im}(T)$.

COROLLARY 1. *If* T *is an injective linear mapping of* X *to* Y, *then the families* $\tilde{\mathbf{q}}$ *and* $\mathbf{q}$ *coincide and so* $\mathbf{q}$ *is a basis of* X.

Let $T \in \operatorname{Hom}_F(X, Y)$. Applying Definition 1 and Lemma 1 with $G = \operatorname{im}(T)$, there exists a basis $(\varepsilon_i)_{i\in I\setminus S}$ of $\operatorname{im}(T)$ such that $\varepsilon_i = \sum_{k\preceq i} \alpha_{ik} \mathbf{b}_k$, with $\alpha_{ii} \neq 0$. If we replace a Noether basis $\mathbf{v}$ with the basis ε of $\operatorname{im}(T)$, a similar procedure, that has been used for the generation of $\mathbf{q}$, results in a family of preimage vectors $\mathbf{e} = (\mathbf{e}_i)_{i\in I\setminus S}$, which satisfy the equation

$$T(\mathbf{e}_i) = \varepsilon_i, \tag{12}$$

for all $i \in I \setminus S$, termed standard family associated with T relative to $\mathbf{b}$. Let $\mathbf{u}$ be a basis of $\operatorname{Ker}(T)$, we then define the extended family $\tilde{\mathbf{e}} = \mathbf{u} \cup \mathbf{e}$ whose indexing set K is the disjoint union of W, $I \setminus S$ equipped with the well order $<_K$, as previously defined in connection with $\tilde{\mathbf{q}}$. Taking into account Remark 3, similar arguments, as those used in Theorem 4, show the following result.

THEOREM 5. *The family $\tilde{\mathbf{e}}$ is a basis of X, called standard basis associated with T relative to* $\mathbf{b}$.

A standard family $\mathbf{e} = (\mathbf{e}_i)_{i\in I\setminus S}$, that satisfies (12), would serve to generalize Ortiz' recursive formula connecting Lanczos' cosets simultaneously with the generation of a recursive formula connecting residual vectors.

THEOREM 6. RECURRENCE. *Let $\prec$ be a well order on I and $\mathbf{e} = (\mathbf{e}_i)_{i\in I\setminus S}$ be a family of vectors satisfying (12). The family of Lanczos' cosets and the family of residual vectors are defined by recursive relations of the form*

$$\mathcal{L}_i = \frac{1}{\alpha_{ii}}\left\{[\mathbf{e}_i] - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathcal{L}_k\right\}, \tag{13}$$

where $[\mathbf{e}_i] = \mathbf{e}_i + \operatorname{Ker}(T)$, and

$$\mathbf{r}_i = \frac{1}{\alpha_{ii}}\left\{\sum_{\substack{k\prec i\\ k\in S}} \alpha_{ik}\mathbf{b}_k - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathbf{r}_k\right\}, \tag{14}$$

for all $i \in I \setminus S$.

PROOF. Let $\tilde{T}$ be the canonical isomorphism induced by T. Let us also call $\Im$ the right-hand side of (13). Applying $\tilde{T}$ to $\Im$, we get

$$\begin{aligned}
\tilde{T}(\Im) &= \tilde{T}\left(\frac{1}{\alpha_{ii}}\left\{[\mathbf{e}_i] - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathcal{L}_k\right\}\right) = \frac{1}{\alpha_{ii}}\left\{\varepsilon_i - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\tilde{T}(\mathcal{L}_k)\right\}\\
&= \frac{1}{\alpha_{ii}}\left\{\sum_{k\preceq i} \alpha_{ik}\mathbf{b}_k - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}(\mathbf{b}_k + \mathbf{r}_k)\right\}\\
&= \mathbf{b}_i + \frac{1}{\alpha_{ii}}\sum_{k\prec i} \alpha_{ik}\mathbf{b}_k - \frac{1}{\alpha_{ii}}\sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathbf{b}_k - \frac{1}{\alpha_{ii}}\sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathbf{r}_k\\
&= \mathbf{b}_i + \frac{1}{\alpha_{ii}}\sum_{\substack{k\prec i\\ k\in S}} \alpha_{ik}\mathbf{b}_k - \frac{1}{\alpha_{ii}}\sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathbf{r}_k.
\end{aligned}$$

Thus,

$$\mathbf{b}_i = \tilde{T}(\Im) - \frac{1}{\alpha_{ii}}\left(\sum_{\substack{k\prec i\\ k\in S}} \alpha_{ik}\mathbf{b}_k - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathbf{r}_k\right), \tag{15}$$

for all $i \in I \setminus S$. As $\tilde{T}(\mathcal{L}_i) = T(\mathbf{q}_i)$ for $i \in I \setminus S$, relation (7) implies

$$\mathbf{b}_i = \tilde{T}(\mathcal{L}_i) - r_i, \tag{16}$$

for all $i \in I \setminus S$. Equating the right-hand sides of (15) and (16), it follows that

$$\tilde{T}(\mathcal{L}_i) - \tilde{T}(\Im) = r_i - \frac{1}{\alpha_{ii}}\left\{\sum_{\substack{k\prec i\\ k\in S}} \alpha_{ik}\mathbf{b}_k - \sum_{\substack{k\prec i\\ k\notin S}} \alpha_{ik}\mathbf{r}_k\right\}. \tag{17}$$

The left-hand side of (17) belongs to $\mathrm{im}(T)$, and the right-hand side belongs to $\mathcal{R}_S$. Since $\mathrm{im}(T) \cap \mathcal{R}_S = \{0\}$, the left-hand side and the right-hand side of (17) must be zero. As $\tilde{T}$ is an isomorphism, (13) and (14) follow. ∎

Let us notice here that the limitations of the indices in the sums of the relations (13) and (14), namely, $k \prec i$, $k \in S$ and $k \prec i$, $k \notin S$, can be replaced by the equivalent notations $k \in I(i) \cap S$ and $k \in I(i) \setminus S$, respectively. Moreover, the formulae (13) and (14) are self-starting recursive relations. In particular, if i_0 is the first element of the set $I \setminus S$, then the recursive formulae imply the relations

$$\mathcal{L}_{i_0} = \frac{1}{\alpha_{i_0 i_0}} [\mathbf{e}_{i_0}], \qquad \mathbf{r}_{i_0} = \frac{1}{\alpha_{i_0 i_0}} \sum_{k \in I(i) \cap S} \alpha_{i_0 k} \mathbf{b}_k.$$

The following corollary is an immediate consequence of Theorem 6.

COROLLARY 2. *Ortiz canonical vectors are defined by a recursive relation of the form*

$$\mathbf{q}_i = \frac{1}{\alpha_{ii}} \left\{ \mathbf{e}_i - \sum_{k \in I(i) \setminus S} \alpha_{ik} \mathbf{q}_k \right\} \tag{18}$$

plus an arbitrary linear combination of elements of $\mathrm{Ker}(T)$.

2.3. Existence and Representation of Solutions to an Abstract Linear Equation

The linear problems of algebra and analysis are concerned with linear mappings on various linear spaces. We mention two kinds of problems: *existence* problems and *representation* problems. Let us suppose that $T \in \mathrm{Hom}_F(X, Y)$. Then we can ask, "For which elements $\mathbf{y}$ in Y does there exist in X an element $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{y}$?" This is the same as asking, "What is the parametric form of the scalar-coordinates of an element $\mathbf{y} \in Y$, relative to $\mathbf{b}$, which belongs to $\mathrm{im}(T)$?" In addition to existence problems, there are representation problems. Given an Ortiz canonical basis of X and some fixed $\mathbf{y} \in \mathrm{im}(T)$, we may ask, "What is the form of the scalar-coordinates of a solution $\mathbf{x} \in X$ to the equation $T(\mathbf{x}) = \mathbf{y}$ expressed in terms of the selected basis?" The following theorem answers these kinds of problems in very general terms.

THEOREM 7. EXISTENCE-REPRESENTATION. *Let* $\mathbf{y} = \sum_{i \in I} \alpha_i \mathbf{b}_i$ *be an arbitrary element of* Y.

(i) $\mathbf{y}$ *belongs to the range of* T *if and only if*

$$\sum_{i \in S} \alpha_i \mathbf{b}_i = \sum_{i \in I \setminus S} \alpha_i \mathbf{r}_i, \tag{19}$$

where $(\mathbf{r}_i)_{i \in I \setminus S}$ *is a family of residual vectors associated with* T, *relative to* $\mathbf{b}$.

(ii) *A solution* $\mathbf{x}$ *of (1) is of the form*

$$\mathbf{x} = \sum_{i \in I \setminus S} \alpha_i \mathbf{q}_i, \tag{20}$$

where $(\mathbf{q}_i)_{i \in I \setminus S}$ *is a family of Ortiz canonical vectors associated with* T, *relative to* $\mathbf{b}$.

PROOF.

(i) Let $\mathbf{q} = (\mathbf{q}_i)_{i \in I \setminus S}$ be a family of Ortiz canonical vectors associated with T, relative to $\mathbf{b}$. In accordance with (6), the family $(T(\mathbf{q}_i))_{i \in I \setminus S}$ is a Noether basis of $\mathrm{im}(T)$ and so $\mathbf{y} \in \mathrm{im}(T)$ if and only if

$$\sum_{i \in I} \alpha_i \mathbf{b}_i = \sum_{i \in I \setminus S} c_i T(\mathbf{q}_i). \tag{21}$$

If we replace in the left-hand side of (21) the expression $\sum_{i\in I}\alpha_i\mathbf{b}_i$ by $\sum_{i\in S}\alpha_i\mathbf{b}_i + \sum_{i\in I\setminus S}\alpha_i\mathbf{b}_i$, and taking into account that $\mathbf{b}_i = T(\mathbf{q}_i) - \mathbf{r}_i$, for $i \in I\setminus S$, then after suitable algebraic manipulations, (21) takes the equivalent form

$$\sum_{i\in S}\alpha_i\mathbf{b}_i - \sum_{i\in I\setminus S}\alpha_i\mathbf{r}_i = \sum_{i\in I\setminus S}(c_i-\alpha_i)T(\mathbf{q}_i). \tag{22}$$

As the left-hand side of (22) belongs to $\mathcal{R}_S$ and the right-hand side belongs to $\mathrm{im}(T)$, it follows from Lemma 1 that both sides are equal to zero. Thus, $c_i = \alpha_i$ for all $i \in I\setminus S$, and $\sum_{i\in S}\alpha_i\mathbf{b}_i - \sum_{i\in I\setminus S}\alpha_i\mathbf{r}_i = 0$. Therefore, $\mathbf{y}\in\mathrm{im}(T)$ if and only if (19) holds true.

(ii) As $c_i = \alpha_i$ for all $i\in I\setminus S$, (21) takes the form $\mathbf{y} = \sum_{i\in I\setminus S}\alpha_i T(\mathbf{q}_i)$ and so $\mathbf{x} = \sum_{i\in I\setminus S}\alpha_i\mathbf{q}_i$ satisfies (1). ∎

2.4. Examples

Let us apply the approach, discussed in this section, to the recursive formulation of the solution for two concrete examples.

EXAMPLE 1. Let $A:\mathbf{R}^4\mapsto\mathbf{R}^6$ be determined by the matrix

$$[A]^{\mathbf{b}}_{\chi} = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 & 0\\ 1 & 3 & 5 & 0 & 0 & 0\\ 2 & 3 & 1 & 6 & 7 & 0\\ 1 & 1 & 2 & 1 & 0 & 6\end{pmatrix}$$

relative to the bases $\chi = (\chi_1,\chi_2,\chi_3,\chi_4)$ and $\mathbf{b} = (\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4,\mathbf{b}_5,\mathbf{b}_6)$ of $\mathbf{R}^4,\mathbf{R}^6$, respectively. Using the convention of row vectors, a row vector $\xi = (\xi_1,\xi_2,\xi_3,\xi_4)$, relative to χ, postmultiplied by the matrix $[A]^{\mathbf{b}}_{\chi}$ results in a row vector $\eta = (\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6)$, relative to $\mathbf{b}$, namely,

$$\eta = \xi[A]^{\mathbf{b}}_{\chi}.$$

Equivalently, we may use the functional notation

$$\sum_i \eta_i\mathbf{b}_i = \sum_k \xi_k A(\chi_k).$$

Assuming that the indexing set $I = \{1,2,3,4,5,6\}$ of $\mathbf{b}$ is equipped with the natural order, it follows from Definition 1 that the set of nonaccessible indices is $S = \{1,4\}$, and therefore, $\mathcal{R}_S = \mathrm{span}\{\mathbf{b}_1,\mathbf{b}_4\}$. Taking into account that the set $I\setminus S = \{2,3,5,6\}$, we define the following list: $\varepsilon_2 = A(\chi_1)$, $\varepsilon_3 = A(\chi_2)$, $\varepsilon_5 = A(\chi_3)$, $\varepsilon_6 = A(\chi_4)$. Thus, equation (12) defines the family $\mathbf{e}$ as follows: $\mathbf{e}_2 = \chi_1$, $\mathbf{e}_3 = \chi_2$, $\mathbf{e}_5 = \chi_3$, $\mathbf{e}_6 = \chi_4$. Since $\mathrm{Ker}(A) = \{0\}$, it follows that $\tilde{\mathbf{e}} = \mathbf{e}$. The Ortiz canonical and residual vectors are defined[4] recursively by (18) of Corollary 2 and (14) of Theorem 6, respectively,

$$\begin{aligned}
\mathbf{q}_2 &= \frac{1}{3}\chi_1,\\
\mathbf{q}_3 &= \frac{1}{5}(\chi_2 - 3\mathbf{q}_2) = \frac{1}{5}(\chi_2-\chi_1),\\
\mathbf{q}_5 &= \frac{1}{7}(\chi_3 - \mathbf{q}_3 - 3\mathbf{q}_2) = \frac{1}{7}\left(\chi_3 - \frac{1}{5}\chi_2 - \frac{4}{5}\chi_1\right),\\
\mathbf{q}_6 &= \frac{1}{6}(\chi_4 - 2\mathbf{q}_3 + \mathbf{q}_2) = \frac{1}{6}\left(\chi_4 - \frac{2}{5}\chi_2 + \frac{1}{15}\chi_1\right),
\end{aligned}$$

[4] Ortiz canonical vectors and residual vectors could be determined by (7). For example, as $A((1/3)\chi_1) = (2/3)\mathbf{b}_1 + \mathbf{b}_2$, it follows that $\mathbf{q}_2 = (1/3)\chi_1$ and $\mathbf{r}_2 = (2/3)\mathbf{b}_1$.

$$\begin{aligned}
\mathbf{r}_2 &= \frac{2}{3}\mathbf{b}_1,\\
\mathbf{r}_3 &= \frac{1}{5}(\mathbf{b}_1 - 3\mathbf{r}_2) = -\frac{1}{5}\mathbf{b}_1,\\
\mathbf{r}_5 &= \frac{1}{7}(\mathbf{b}_1 + 6\mathbf{b}_4 - 3\mathbf{r}_2 - \mathbf{r}_3) = \frac{1}{7}\left(\frac{1}{5}\mathbf{b}_1 + 6\mathbf{b}_4\right),\\
\mathbf{r}_6 &= \frac{1}{6}(\mathbf{b}_1 + \mathbf{b}_4 - \mathbf{r}_2 - 2\mathbf{r}_3) = \frac{1}{6}\left(\frac{11}{15}\mathbf{b}_1 + \mathbf{b}_4\right).
\end{aligned}$$

Let us consider the equation

$$A(\mathbf{x}) = \mathbf{y}.$$

The parametric form of the scalar-coordinates of a right-hand side vector $\mathbf{y} = \alpha\mathbf{b}_1 + c\mathbf{b}_2 + d\mathbf{b}_3 + \beta\mathbf{b}_4 + f\mathbf{b}_5 + g\mathbf{b}_6$ can be found with the use of (19) of Theorem 7. Thus, $\alpha\mathbf{b}_1 + \beta\mathbf{b}_4 = c\mathbf{r}_2 + d\mathbf{r}_3 + f\mathbf{r}_5 + g\mathbf{r}_6$, and using the explicit form of $\mathbf{r}_i$, we have

$$\alpha\mathbf{b}_1 + \beta\mathbf{b}_4 = \frac{2c}{3}\mathbf{b}_1 - \frac{d}{5}\mathbf{b}_1 + \frac{f}{7}\left(\frac{1}{5}\mathbf{b}_1 + 6\mathbf{b}_4\right) + \frac{g}{6}\left(\frac{11}{15}\mathbf{b}_1 + \mathbf{b}_4\right).$$

Hence, $\alpha = (2/3)c - (1/5)d + (1/35)f + (11/90)g$ and $\beta = (6/7)f + (1/6)g$, and so the right-hand side vector must be of the form

$$\mathbf{y} = \left(\frac{2c}{3} - \frac{d}{5} + \frac{f}{35} + \frac{11g}{90}\right)\mathbf{b}_1 + c\mathbf{b}_2 + d\mathbf{b}_3 + \left(\frac{6f}{7} + \frac{g}{6}\right)\mathbf{b}_4 + f\mathbf{b}_5 + g\mathbf{b}_6,$$

where c, d, f, g are free scalars. Finally, a solution $\mathbf{x}$ of the above equation, expressed in terms of Ortiz canonical vectors, is given by

$$\mathbf{x} = c\mathbf{q}_2 + d\mathbf{q}_3 + f\mathbf{q}_5 + g\mathbf{q}_6.$$

Using the explicit form of $\mathbf{q}_i$, the parametric form of the solution takes the form

$$\mathbf{x} = \frac{c}{3}\chi_1 + \frac{d}{5}(\chi_2 - \chi_1) + \frac{f}{7}\left(\chi_3 - \frac{1}{5}\chi_2 - \frac{4}{5}\chi_1\right) + \frac{g}{6}\left(\chi_4 - \frac{2}{5}\chi_2 + \frac{1}{15}\chi_1\right).$$

Let us now consider the order relation induced on I by the binary relation $\prec_* = \{(2,1),(2,3),(2,4),(2,5),(2,6),(1,3),(1,4),(1,5),(1,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$. The elements of I can be listed in this order as follows: $2 \prec_* 1 \prec_* 3 \prec_* 4 \prec_* 5 \prec_* 6$. Thus, the set of nonaccessible indices is $S = \{2,4\}$, and so $\mathcal{R}_S = \operatorname{span}\{\mathbf{b}_2, \mathbf{b}_4\}$. Now the new lists of Ortiz canonical and residual vectors are

$$\begin{aligned}
\mathbf{q}_1 &= \frac{1}{2}\chi_1, & \mathbf{r}_1 &= \frac{3}{2}\mathbf{b}_2,\\
\mathbf{q}_3 &= \frac{1}{5}\left(\chi_2 - \frac{1}{2}\chi_1\right), & \mathbf{r}_3 &= \frac{3}{10}\mathbf{b}_2,\\
\mathbf{q}_5 &= \frac{1}{7}\left(\chi_3 - \frac{1}{5}\chi_2 - \frac{9}{10}\chi_1\right), & \mathbf{r}_5 &= \frac{1}{7}\left(-\frac{3}{10}\mathbf{b}_2 + 6\mathbf{b}_4\right),\\
\mathbf{q}_6 &= \frac{1}{6}\left(\chi_4 - \frac{2}{5}\chi_2 - \frac{3}{10}\chi_1\right), & \mathbf{r}_6 &= \frac{1}{6}\left(-\frac{11}{10}\mathbf{b}_2 + \mathbf{b}_4\right).
\end{aligned}$$

Let us point out here that $\operatorname{Ker}(T) = \{0\}$ and that the set of nonaccessible indices determined by the order $\prec_*$ differs from the set of nonaccessible indices determined by the natural order. Accordingly, the sets of nonaccessible indices generate different lists of Ortiz canonical and residual vectors, as indicated above. These lists contain Ortiz canonical vectors of the same index for

$i = 3, 5, 6$. However, they do not differ by an element of $\mathrm{Ker}(T)$, as we have already mentioned in Remark 3.

Before considering the second example, let us define the following well-order relations on the set $I = \mathbf{N}_0 \times \mathbf{N}_0$, where the set $\mathbf{N}_0$ denotes the set of natural numbers including zero.

(i) By $\prec_{C1}$ we shall denote the order relation, induced by Cantor's first diagonal method, as it is displayed below.

$$\begin{array}{lllll}
(0,0) & (0,1) & (0,2) & (0,3) & \cdots \\
 & \nearrow & \nearrow & \nearrow & \\
(1,0) & (1,1) & (1,2) & \cdots & \\
 & \nearrow & \nearrow & & \\
(2,0) & (2,1) & \cdots & & \\
 & \nearrow & & & \\
(3,0) & \cdots & & & \\
\cdots & & & &
\end{array}$$

The elements of I can be listed in this order as follows: $(0,0)\prec_{C1}(1,0)\prec_{C1}(0,1)\prec_{C1}(2,0)\prec_{C1}(1,1)\prec_{C1}(0,2)\prec_{C1}(3,0)\prec_{C1}(2,1)\prec_{C1}(1,2)\prec_{C1}(0,3)\prec_{C1},\ldots$. The ordinal number corresponding to the ordered set $(I, \prec_{C1})$ is ω.

(ii) In a similar manner, we can define the well-order relation $\prec_{C2}$, as follows:

$$\begin{array}{lllll}
(0,0) & (0,1) & (0,2) & (0,3) & \cdots \\
 & \swarrow & \swarrow & \swarrow & \\
(1,0) & (1,1) & (1,2) & \cdots & \\
 & \swarrow & \swarrow & & \\
(2,0) & (2,1) & \cdots & & \\
 & \swarrow & & & \\
(3,0) & \cdots & & & \\
\cdots & & & &
\end{array}$$

The elements of I can be listed in this order as follows: $(0,0)\prec_{C2}(0,1)\prec_{C2}(1,0)\prec_{C2}(0,2)\prec_{C2}(1,1)\prec_{C2}(2,0)\prec_{C2}(0,3)\prec_{C2}(1,2)\prec_{C2}(2,1)\prec_{C2}(3,0)\prec_{C2},\ldots$. The ordinal number corresponding to the ordered set $(I, \prec_{C2})$ is also ω.

EXAMPLE 2. Let us consider Laplace's operator restricted to the space of real bivariate polynomials

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let us also consider the standard basis $(x^i y^j)_{(i,j)\in I}$ of bivariate polynomials, whose indexing set is well ordered by $\prec_{C1}$. The range of ∇^2 is generated by polynomials of the form

$$\nabla^2 \left(x^n y^m\right) = n(n-1)x^{n-2}y^m + m(m-1)x^n y^{m-2}. \tag{23}$$

Taking into account that $(n, m-2) \prec_{C_1} (n-2, m)$, it follows from (23) that

$$\operatorname{maxsupp}\left(\nabla^2\left(x^n y^m\right)\right) = (n-2, m), \qquad \text{if } n \geq 2 \quad \text{and} \quad m \in \mathbf{N}_0.$$

We can also write $\operatorname{maxsupp}(\nabla^2(x^{n+2}y^m)) = (n, m)$ for all $n, m \in \mathbf{N}_0$, which shows that $S = \emptyset$. Thus, the residual space is $\mathcal{R}_S = \{0\}$ and ∇^2 is an epimorphism. Following the notation of

Theorem 5, after the identification of $\varepsilon_{n-2\,m}$ with $n(n-1)x^{n-2}y^m+m(m-1)x^ny^{m-2}$ and $\mathbf{e}_{n-2\,m}$ with x^ny^m for $n \geq 2$, $m \in \mathbf{N}_0$, the family $(\mathbf{e}_{n-2\,m})_{n\geq 2, m\in\mathbf{N}_0}$ solves the equation $\nabla^2(\mathbf{e}_{n-2\,m}) = \varepsilon_{n-2\,m}$. The former corresponds to equation (12), and therefore, the recursive relation of the Ortiz canonical polynomials, given by (13), takes the form

$$Q^{(1)}_{n-2\,m}(x,y) = \frac{1}{n(n-1)}\left\{x^ny^m - m(m-1)Q^{(1)}_{n\,m-2}(x,y)\right\}, \tag{24}$$

for $n \geq 2$ and $m \in \mathbf{N}_0$. Let us derive some more Ortiz canonical bivariate polynomials generated by (24) and written in ascending order, relative to $\prec_{C1}$,

$$\begin{aligned}
Q^{(1)}_{00}(x,y) &= \frac{1}{2}x^2, && \text{for } n=2, \quad m=0,\\
Q^{(1)}_{10}(x,y) &= \frac{1}{6}x^3, && \text{for } n=3, \quad m=0,\\
Q^{(1)}_{01}(x,y) &= \frac{1}{2}x^2y, && \text{for } n=2, \quad m=1,\\
Q^{(1)}_{20}(x,y) &= \frac{1}{12}x^4, && \text{for } n=4, \quad m=0,\\
Q^{(1)}_{11}(x,y) &= \frac{1}{6}x^3y, && \text{for } n=3, \quad m=1,\\
Q^{(1)}_{02}(x,y) &= \frac{1}{2}\left\{x^2y^2 - \frac{1}{6}x^4\right\}, && \text{for } n=2, \quad m=2,\\
&\vdots && \qquad\vdots
\end{aligned}$$

The bivariate polynomial solution of the differential equation

$$\nabla^2 t(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\alpha_{ij}x^iy^j$$

is given by

$$t(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\alpha_{ij}Q^{(1)}_{ij}(x,y)$$

plus an arbitrary linear combination of elements of $\operatorname{Ker}(\nabla^2)$.

Let us now consider the sequence of the Ortiz canonical polynomials generated by the order $\prec_{C2}$, defined on the indexing set I of $(x^iy^j)_{(i,j)\in I}$. Working similarly as in the previous case, it follows from (23) that

$$\operatorname{maxsupp}\left(\nabla^2\left(x^ny^m\right)\right) = (n, m-2), \qquad \text{if } m \geq 2 \quad \text{and} \quad n \in \mathbf{N}_0,$$

and therefore,

$$Q^{(2)}_{n\,m-2}(x,y) = \frac{1}{m(m-1)}\left\{x^ny^m - n(n-1)Q^{(2)}_{n-2\,m}(x,y)\right\}, \tag{25}$$

for $m \geq 2$ and $n \in \mathbf{N}_0$. The Ortiz canonical bivariate polynomials written in ascending order, relative to $\prec_{C2}$, are

$$\begin{aligned}
Q^{(2)}_{00}(x,y) &= \frac{1}{2}y^2, && \text{for } m=2, \quad n=0,\\
Q^{(2)}_{01}(x,y) &= \frac{1}{6}y^3, && \text{for } m=3, \quad n=0,
\end{aligned}$$

$$\begin{aligned}
Q_{10}^{(2)}(x,y) &= \frac{1}{2}xy^2, && \text{for } m=2, \quad n=1,\\
Q_{02}^{(2)}(x,y) &= \frac{1}{12}y^4, && \text{for } m=4, \quad n=0,\\
Q_{11}^{(2)}(x,y) &= \frac{1}{6}xy^3, && \text{for } m=3, \quad n=1,\\
Q_{20}^{(2)}(x,y) &= \frac{1}{2}\{x^2y^2-\frac{1}{6}y^4\}, && \text{for } n=2, \quad m=2,\\
&\vdots && \vdots
\end{aligned}$$

As an application of Theorem 2, the elements of $\mathrm{Ker}(\nabla^2)$ may be generated with the use of the above-mentioned sequences of Ortiz canonical polynomials. Two Ortiz canonical polynomials $Q_{nm}^{(1)}(x,y)$, $Q_{nm}^{(2)}(x,y)$ of the same index differ by an element of $\mathrm{Ker}(\nabla^2)$, since the above-defined orderings of I result in the same $S=\emptyset$. In particular, the set of bivariate polynomials generated by the differences $u_{nm}(x,y)=Q_{nm}^{(1)}(x,y)-Q_{nm}^{(2)}(x,y)$ and extended by the set $\{1,x,y,xy\}$, whose elements are mapped to zero directly by (23), form a generating system of the space $\mathrm{Ker}(\nabla^2)$ of harmonic[5] polynomials. Moreover, the differences $u_{nm}(x,y)$ result in either linearly independent harmonic polynomials such as

$$\begin{aligned}
Q_{10}^{(1)}(x,y)-Q_{10}^{(2)}(x,y) &= \frac{1}{6}x^3-\frac{1}{2}xy^2,\\
Q_{01}^{(1)}(x,y)-Q_{01}^{(2)}(x,y) &= \frac{1}{2}x^2y-\frac{1}{6}y^3,
\end{aligned}$$

or linearly dependent harmonic polynomials such as

$$\begin{aligned}
Q_{02}^{(1)}(x,y)-Q_{02}^{(2)}(x,y) &= \frac{1}{2}x^2y^2-\frac{1}{12}x^4-\frac{1}{12}y^4,\\
Q_{20}^{(1)}(x,y)-Q_{20}^{(2)}(x,y) &= \frac{1}{12}x^4-\frac{1}{2}x^2y^2+\frac{1}{12}y^4.
\end{aligned}$$

3. INFINITE MATRICES IN ROW ECHELON FORM ASSOCIATED WITH ABSTRACT LINEAR MAPPINGS

Throughout this section, we adopt the following assumptions and notations: X and Y are vector spaces equipped with ordered bases $\chi=(\chi_k)_{k\in K}$ and $\mathbf{b}=(\mathbf{b}_i)_{i\in I}$ over well-ordered indexing sets $(K,\prec_K)$, $(I,\prec_I)$, respectively; $A\in \mathrm{Hom}_F(X,Y)$ and is defined by $A(\chi_k)=\sum_{i\in I}\alpha_{ki}\mathbf{b}_i$ with $\alpha_{ki}\in F$; $[A]=(\alpha_{ki})_{(k,i)\in K\times I}$ stands for the matrix associated with A, relative to χ and $\mathbf{b}$; W denotes a subset of K defined by: $k\in W \Leftrightarrow A(\chi_k)=0$; J stands for the set complement of W; σ_j denotes the maxsupp$(A(\chi_j))$ for $j\in J$, relative to $(J,\prec_{I\setminus S})$. We shall also write $A(\chi_j)=\sum_{i\preceq_I\sigma_j}\alpha_{ji}\mathbf{b}_i$ for $j\in J$, with $\alpha_{j\sigma_j}\neq 0$. As the number of nonzero entries, in any row of $[A]$, is finite, such infinite matrices are naturally called *row finite*. The class of row finite matrices is in one-to-one correspondence [12, pp. 243–244] with the class $\mathrm{Hom}_F(X,Y)$ relative to χ and $\mathbf{b}$. An order monomorphism f of $(K,\prec_K)$ into $(I,\prec_I)$ is a sharp order preserving mapping; namely, $k\prec_K m$ implies $f(k)\prec_I f(m)$ for all $k,m\in K$. It follows that f is an injective (or one-to-one) mapping.

The definition below extends the notion of row echelon finite matrices to cover the case of infinite matrices over well-ordered indexing sets.

DEFINITION 2. *A row finite matrix $[A]=(\alpha_{ki})_{(k,i)\in K\times I}$ is said to be in row echelon (RE) form, relative to $(K,\prec_K)$, $(I,\prec_I)$ if and only if the following hold.*

(i) $\alpha_{j\sigma_j}=1$, *for all $j\in J$ (the greatest element of the support of any nonzero row of index $j\in J$, is the scalar 1).*

[5]These are the polynomial solutions of the equation $\nabla^2 u_{nm}(x,y)=0$.

(ii) *If $\alpha_{ki} = 0$ for some $k \in K$ and for all $i \in I$, then $\alpha_{mi} = 0$ for all $m \prec_K k$ and for all $i \in I$ (all the predecessors of a zero row are zero rows).*

(iii) *The mapping $\sigma : J \mapsto I$ defined by $\sigma(j) = \sigma_j$ is an order monomorphism (if $k \in J$ and $m \in J$ such that $k \prec_K m$, then $\sigma_k \prec_I \sigma_m$).*

A row finite matrix $[A]$, satisfying postulates (ii) and (iii) with $\alpha_{j\sigma_j} \neq 0$ for $j \in J$, is said to be in *prerow echelon form*.

3.1. Detecting Matrices in Row Echelon Form

In the general case of row finite matrices, the set $\sigma(J) = \{\sigma_j : j \in J\}$ is a subset of $I \setminus S$, but it is not necessarily equal to $I \setminus S$; however, in the context of matrices in RE form the equality of these sets holds true. The foregoing result together with some additional properties of matrices in RE form are demonstrated in the following lemma.

LEMMA 2. *Let $[A] = (\alpha_{ki})_{(k,i)\in K\times I}$ be in prerow echelon form.*

(i) *The set of nonaccessible indices S is set complement of $\sigma(J)$, and therefore,*

$$\sigma(J) = I \setminus S.$$

(ii) *The family $(A(\chi_j))_{j\in J}$ is a basis of $\operatorname{im}(A)$.*

(iii) *The family $(\chi_w)_{w\in W}$ is a basis of $\operatorname{Ker}(A)$.*

(iv) *Every predecessor $\alpha_{i\sigma_j}$ of a leading one $\alpha_{j\sigma_j}$ is zero.*

PROOF.

(i) It suffices to show that $I \setminus S \subset \sigma(J)$. If $i \in I \setminus S$, it follows from Definition 1 that there exists some $\mathbf{g} \in \operatorname{im}(A)$ such that $\operatorname{maxsupp}(\mathbf{g}) = i$ with $\mathbf{g} \neq 0$. Expressing $\mathbf{g}$ in terms of $(A(\chi_j))_{j\in J}$, it takes the form $\mathbf{g} = \sum_{j \preceq_K k} \mu_j A(\chi_j)$, for $\mu_j \neq 0$. Since σ is an order monomorphism, it follows that $\sigma_j \prec_I \sigma_k$ for all $j \in J$ such that $j \prec_K k$. The former means that $\sigma_k = \operatorname{maxsupp}(\mathbf{g})$ and so $i = \sigma_k$. Thus, $i \in \sigma(J)$ and the assertion follows.

(ii) By virtue of Definition 2, the mapping $\sigma : J \mapsto I$ is injective, namely, $k \neq m \Leftrightarrow \sigma_k \neq \sigma_m$; thus, we can define $\boldsymbol{\varepsilon}_{\sigma_j} = A(\chi_j)$ for all $j \in J$. As $A(\chi_j) = \sum_{i \preceq_I \sigma_j} \alpha_{ji}\mathbf{b}_i$ for $\alpha_{j\sigma_j} \neq 0$, we have $\operatorname{maxsupp}(\boldsymbol{\varepsilon}_{\sigma_j}) = \sigma_j$ for all $j \in J$. As $\sigma(J) = I \setminus S$ we can also write $\operatorname{maxsupp}(\boldsymbol{\varepsilon}_i) = i$ for all $i \in I \setminus S$. By virtue of Lemma 1(ii), the family $(\boldsymbol{\varepsilon}_i)_{i\in I\setminus S}$ or $(\boldsymbol{\varepsilon}_{\sigma_j})_{j\in J}$ is a basis of $\operatorname{im}(A)$ and the result follows.

(iii) It suffices to show that $(\chi_w)_{w\in W}$ is a generating system of $\operatorname{Ker}(A)$. Let us take any $\mathbf{x} \in \operatorname{Ker}(A)$. Since W, J are complementary sets, we can write $\mathbf{x} = \sum_{w\in W} \alpha_w \chi_w + \sum_{j\in J} \alpha_j \chi_j$. Now the linear independence of $(A(\chi_j))_{j\in J}$ implies the equivalences, $A(\mathbf{x}) = 0 \Leftrightarrow \sum_{j\in J} \alpha_j A(\mathbf{x}_j) = 0 \Leftrightarrow \alpha_j = 0$ for all $j \in J$. Thus, $\mathbf{x} = \sum_{w\in W} \alpha_w \chi_w$ and the assertion follows.

(iv) If $i \in W$, then the row of index i is a zero row and the result follows. If $i \in J$ and $i \prec_K j$, then the scalar $\alpha_{i\sigma_i}$ is a leading one. Taking into account that σ is an order monomorphism, we have $\sigma_i \prec_I \sigma_j$. Thus, $\alpha_{i\sigma_j} = 0$. ∎

In order to formulate a useful criterion for the detection of matrices in RE form, we introduce some additional definitions and notations to those stated in the beginning of this section. Let $[A] = (\alpha_{ki})_{(k,i)\in K\times I}$ be a row finite matrix associated with A, relative to the bases $\chi = (\chi_k)_{k\in K}$, $\mathbf{b} = (\mathbf{b}_i)_{i\in I}$. Let also $\prec_I$ be a well-ordering of I and $\prec_W$ be the well-ordering of W induced by $\prec_K$. If the mapping $\sigma : J \mapsto I$ is injective, we then define on J a well-order relation $\prec_{\sigma^{-1}}$ induced by the inverse mapping of σ, namely, $\sigma^{-1}(m) \prec_{\sigma^{-1}} \sigma^{-1}(n)$ if and only if $m \prec_I n$ for $m, n \in \sigma(J)$. Moreover, we define on K the order relation $<_K \; = \; \prec_W \cup \prec_{\sigma^{-1}} \cup (W \times J)$. Formally, $(K, <_K)$ is a well-ordered set. The mapping σ is an order monomorphism and the elements of W are predecessors of the elements of J, relative to $<_K$. Let us call $\chi^\dagger$ the family χ indexed by the ordered set $(K, <_K)$. Let us also introduce the family $\chi^* = (\chi_k^*)_{k\in K}$, defined by

$$\chi_k^* = \begin{cases} \chi_k, & \text{if } k \in W, \\ \alpha_{k\sigma_k}^{-1}\chi_k, & \text{if } k \in J, \end{cases} \tag{26}$$

whose indexing set is ordered by $<_K$. Formally χ^* is a basis of X. Moreover, it follows from (26) that the coordinate of index (j, σ_j), in the row determined by $A(\chi_j^*)$, is $\alpha_{j\sigma_j} = 1$ for all $j \in J$. Thus, the following theorem has been established.

THEOREM 8. *If $\sigma : J \mapsto I$ is an injective mapping, then the row finite matrix $[A]$ is in RE form (respectively, pre-RE form), relative to the bases χ^*, $\mathbf{b}$ (respectively, $\chi^\dagger$, $\mathbf{b}$).*

A consequence of Lemma 2 and Theorem 8 is the following corollary.

COROLLARY 3. *If $\sigma : J \mapsto I$ is an injective mapping, then the family $(A(\chi_j))_{j\in J}$ is a basis of $\mathrm{im}(A)$; the family $(\chi_w)_{w\in W}$ is a basis of $\mathrm{Ker}(A)$; and the set S, of nonaccessible indices, is set complement of $\sigma(J)$.*

3.2. Matrices in Row Echelon Form: Existence-Recurrence

Lemma 1 and Theorem 5 guarantee the existence of a standard basis $\tilde{\mathbf{e}} = \mathbf{u} \cup \mathbf{e}$ of the domain space of an arbitrary linear mapping, which is defined implicitly through a basis $\boldsymbol{\varepsilon}$ of $\mathrm{im}(A)$. The indexing set K of $\tilde{\mathbf{e}}$ is the disjoint union of the sets W, $I \setminus S$, which is well ordered by $<_K$. Thus, the set complement J of W of Definition 2 is $I \setminus S$. Adopting the notation of Theorem 5, the following result shows the existence of matrices in RE form associated with abstract linear mappings.

THEOREM 9. *Let $\mathbf{b}$ be any basis of Y whose indexing set I is well ordered by $\prec_I$. Let also $A \in \mathrm{Hom}_F(X, Y)$. Then $[A]$ is in pre-RE form, relative to $\tilde{\mathbf{e}}$, $\mathbf{b}$. Applying (26) with $\chi = \tilde{\mathbf{e}}$, the matrix $[A]$ is in RE form, relative to $\tilde{\mathbf{e}}^*$, $\mathbf{b}$.*

PROOF. From (12), we have $A(\mathbf{e}_i) = \varepsilon_i$ with $\mathrm{maxsupp}(\varepsilon_i) = i$ for all $i \in I \setminus S$. As $J = I \setminus S$, it follows that the mapping $\sigma : J \mapsto J$ is the identity mapping. Hence, the result follows from Theorem 8. ∎

The following corollary shows that the recursive relations (13) and (14) are conveniently generated through pre-RE matrices. Following the notation of Definition 2, the above-mentioned recursive relations can take a useful form in connection with pre-RE matrices.

COROLLARY 4. *If $[A] = (\alpha_{ki})_{(k,i)\in K\times I}$ is in pre-RE form, then the family of Ortiz canonical vectors and the family of residual vectors are defined by recursive relations of the form*

$$\mathbf{q}_{\sigma_j} = \frac{1}{\alpha_{j\sigma_j}} \left\{ \chi_j - \sum_{i\in I_j} \alpha_{ji}\mathbf{q}_i \right\} \tag{27}$$

plus an arbitrary linear combination of elements of $\mathrm{Ker}(A)$ and

$$\mathbf{r}_{\sigma_j} = \frac{1}{\alpha_{j\sigma_j}} \left\{ \sum_{s\in S_j} \alpha_{js}\mathbf{b}_s - \sum_{i\in I_j} \alpha_{ji}\mathbf{r}_i \right\}, \tag{28}$$

for all $j \in J$, where $I_j = I(\sigma_j) \setminus S$ and $S_j = S \cap I(\sigma_j)$.

PROOF. Using similar arguments with those used in the proof of Lemma 2(ii), we can define $\varepsilon_{\sigma_j} = A(\chi_j)$ for all $j \in J$. It also follows from the above-mentioned lemma that the family $(\varepsilon_{\sigma_j})_{j\in J}$ is a basis of $\mathrm{im}(A)$. Now equation (12) takes the form $A(\mathbf{e}_{\sigma_j}) = \varepsilon_{\sigma_j}$ for all $j \in J$. Consequently, $\chi_j \in [\mathbf{e}_{\sigma_j}]$ and the result follows from Theorem 6 and Corollary 2. ∎

The results derived in this section would serve to extend, without essential modifications, the procedure discussed in Example 1 for the treatment of finite systems of linear equations to the case of equations determined by infinite matrices in row echelon form. After the verification of the RE form of an infinite matrix, for which the criterion demonstrated in Theorem 8 could be used, bases of the null and the residual space are directly generated and the recursive relations (27),(28) are immediately constructible (see Examples 3 and 4 in Section 4).

4. APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

In the following examples, a partial differential operator,

$$D = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij}(x, y) \frac{\partial^{i+j}}{\partial x^i \, \partial y^j},$$

with coefficients $p_{ij}(x, y)$ real bivariate polynomials, is restricted to the space $\mathcal{R}[x, y]$ of real bivariate polynomials. As D is linear and maps bivariate polynomials into bivariate polynomials, it follows that D is an endomorphism of $\mathbf{R}[x, y]$. We shall use the recursive approach to construct the parametric form of the right-hand side bivariate polynomials $f(x, y)$ and the exact polynomial solution of a partial differential equation

$$D\, t(x, y) = f(x, y). \tag{29}$$

At this point, let us put our results in the context of the Tau method. Let us assume that a solution $t(x, y)$ to (29) needs to fulfill some supplementary conditions and that the polynomial $f(x, y)$ must be an element of $\operatorname{im}(D)$. Then, following the lines of the Tau method [7,9], a small perturbation term[6] $H(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{K} \tau_{ij} T_{ij}(x, y)$ is added to $f(x, y)$ to satisfy the above requirements. A τ-approximate polynomial solution is the exact solution to the perturbed problem

$$D\, t(x, y) = f(x, y) + H(x, y), \tag{30}$$

which is expressible in terms of Ortiz canonical polynomials by (20).

In addition to the order relations $\prec_{C1}$, $\prec_{C2}$, defined in Section 2, the indexing set $I = \mathbf{N}_0 \times \mathbf{N}_0$ is equipped with the *lexicographic* $\prec_L$ well ordering defined in the sequel. We say that (n, m) is less than (k, l), relative to $\prec_L$, and we write $(n, m) \prec_L (k, l)$ if and only if either $(n < k)$ or $(n = k$ and $m < l)$, where "$<$" stands for the natural order on $\mathbf{N}_0$. The elements of I can be listed in this order[7] as follows: $(0,0) \prec_L (0,1) \prec_L (0,2) \prec_L, \dots, \prec_L (1,0) \prec_L (1,1) \prec_L (1,2) \prec_L, \dots, \prec_L (n,0) \prec_L (n,1) \prec_L (n,2) \prec_L, \dots$. The ordinal number corresponding to the ordered set $(I, \prec_L)$ is ω^2.

Following Hosseini and Ortiz [9], a bivariate polynomial $p_{[nm]}(x, y)$ is called *rectangular*[8] of index (n, m) if it can be written in the form $p_{[nm]}(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{ij} x^i y^j$ with $\alpha_{nm} \neq 0$. If the indices of the nonzero coefficients α_{ij} of a rectangular polynomial are listed in an ascending order, relative to $\prec_L$, then (n, m) occupies the highest entry of the list and so it coincides with $\operatorname{maxsupp}(p_{[nm]}(x, y))$. A bivariate polynomial $p_{\langle nm \rangle}(x, y)$ is called *triangular* of index (n, m), relative to $\prec_{C_1}$ (respectively, $\prec_{C_2}$) if and only if it is of the form

$$p_{\langle nm \rangle}(x, y) = \sum_{\substack{k=0 \\ i+j=k}}^{n+m} \alpha_{ij} x^i y^j$$

[6]It could be a finite linear combination of elements $T_{ij}(x, y)$ chosen from the Chebyshev product basis.

[7]We mention here that the elements $(n, 0)$ for $n \geq 1$ do not have immediate predecessors, relative to the lexicographic order.

[8]A list of indices (i, j) of the coefficients of $p_{[nm]}(x, y)$, written in a two-dimensional array, relative to $\prec_L$, forms a rectangle.

assuming that $\alpha_{nm} \neq 0$ and $\alpha_{ij} = 0$, whenever $i+j = n+m$ with $i < n$ and $j > m$ (respectively, $i > n$ and $j < m$). Let us give a list of the first triangular polynomials, relative to $\prec_{C_1}$,

$$\begin{aligned}
p_{\langle 0,0\rangle}(x,y) &= a_{00},\\
p_{\langle 10\rangle}(x,y) &= a_{00} + a_{10}x,\\
p_{\langle 01\rangle}(x,y) &= a_{00} + a_{10}x + a_{01}y,\\
p_{\langle 20\rangle}(x,y) &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2,\\
p_{\langle 11\rangle}(x,y) &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy,\\
p_{\langle 02\rangle}(x,y) &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2,\\
&\vdots
\end{aligned}$$

If the terms of a triangular polynomial $p_{\langle nm\rangle}(x,y)$ are listed in an ascending order, relative to $\prec_{C1}$ (respectively, relative to $\prec_{C2}$), then the index (n,m) occupies the highest entry of the list among the nonzero coefficients of $p_{\langle nm\rangle}(x,y)$ and so it coincides with $\operatorname{maxsupp}(p_{\langle nm\rangle}(x,y))$.

REMARK 4. By virtue of Lemma 1(ii), applied for $S = \emptyset$, both sequences of rectangular and triangular bivariate polynomials form bases of $\mathbf{R}[x,y]$.

EXAMPLE 3. Let the differential operator be $D = (x^2+1)\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2}$.

Following the notation of Section 3, the operator D applied to $\chi = (x^n y^m)_{(n,m)\in I}$ generates the polynomials

$$D(x^n y^m) = n(n-1)x^{n-2}y^m + m(m-1)x^n y^{m-1} + n(n-1)x^n y^m, \tag{31}$$

which form a generating family of $\operatorname{im}(D)$. If $n \leq 1$, then (31) takes the form

$$D(x^n y^m) = m(m-1)x^n y^{m-1}. \tag{32}$$

It follows from (31) that the indexing set W is $\{(0,0),(0,1),(1,0),(1,1)\}$. The set complement J of W is the union of the disjoint sets $\{(n,m),\ n \geq 2,\ m \in \mathbf{N}_0\}$ and $\{(n,m),\ n \leq 1,\ m \geq 2\}$. Let us consider the basis $\mathbf{b} = (x^n y^m)_{(n,m)\in I}$, of the codomain space $\mathbf{R}[x,y]$ of D, whose indexing set $I = \mathbf{N}_0 \times \mathbf{N}_0$ is well ordered by the lexicographic order $\prec_L$. The elements of $\mathbf{b}$ are listed, relative to $\prec_L$, as $(1, y, y^2, \dots, x, xy, xy^2, \dots, x^2, x^2y, x^2y^2, \dots)$. Taking into account that $(n-2,m) \prec_L (n,m-1) \prec_L (n,m)$, it follows from (31) and (32) that

$$\operatorname{maxsupp}(D(x^n y^m)) = \begin{cases} (n,m), & \text{if } n \geq 2 \text{ and } m \in \mathbf{N}_0,\\ (n,m-1), & \text{if } n \leq 1 \text{ and } m \geq 2. \end{cases} \tag{33}$$

The mapping $\sigma : J \mapsto I$, defined by $\sigma(n,m) = \operatorname{maxsupp}(D(x^n y^m))$, is formally injective. It follows from Corollary 3 that $\operatorname{Ker}(D) = \operatorname{span}\{1, x, y, xy\}$, and that $S = \{(0,0),(1,0)\}$, as being set-complement of $\sigma(J)$. Therefore, the residual space[9] is $\mathcal{R}_S = \operatorname{span}\{1, x\}$. Following the procedure demonstrated in Section 3.1, let us define the order relation induced on J by the inverse of σ. It generates the domain basis $\chi^\dagger = (1, x, y, xy, y^2, y^3, \dots, xy^2, xy^3, \dots, x^2, x^2y, \dots)$.

[9]As $(n-2,m) \prec_{C_1} (n,m-1) \prec_{C_1} (n,m)$ and $(n-2,m) \prec_{C_2} (n,m-1) \prec_{C_2} (n,m)$, it follows that in both cases $\operatorname{maxsupp}(D(x^n y^m))$ is given by (33). Consequently, the orders $\prec_{C_1}$, $\prec_{C_2}$, $\prec_L$ generate the same S. Theorem 2 implies that in all these cases Ortiz canonical polynomials of the same index differ by an element of $\operatorname{Ker}(D)$.

The matrix representation of D, relative to $\chi^{\dagger}$, $\mathbf{b}$, is of the form

$$
\left(\begin{array}{cccc|cccc|cccc|cccc|c}
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
\hline
0 & 2 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 6 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 12\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
 & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots & \cdots \\
\hline
0 & 0 & 0 & 0\cdots & 0 & 2 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 6 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 12\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
 & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots & \cdots \\
\hline
2 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 2 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 2 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 2 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 2 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 2 & 2 & 0\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 2\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 3 & 2\cdots & 0 & 0 & 0 & 0\cdots & \cdots \\
 & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots & \cdots \\
\hline
0 & 0 & 0 & 0\cdots & 6 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 6 & 0 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 6 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 6 & 0 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 6 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 2 & 6 & 0\cdots & \cdots \\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 6\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 & 6 & 6\cdots & \cdots \\
 & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots & \cdots \\
\hline
 & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots & \cdots
\end{array}\right).
$$

The foregoing matrix is in pre-RE form, relative to Definition 2. If $n \leq 1$ and $m \geq 2$, then relations (27) and (28) in conjunction with (32), or directly from (7), generate the Ortiz canonical and residual polynomials[10]

$$Q_{n\,m-1}(x,y) = \frac{1}{m(m-1)} x^n y^m; \qquad R_{n\,m-1}(x,y) = 0. \tag{34}$$

If $n \geq 2$ and $m \in \mathbf{N}_0$, then, by virtue of (27), we get with the aid of (31) the recursive relation of the remaining Ortiz canonical polynomials

$$Q_{nm}(x,y) = \frac{1}{n(n-1)} \left\{ x^n y^m - m(m-1) Q_{nm-1}(x,y) - n(n-1) Q_{n-2\,m}(x,y) \right\}. \tag{35}$$

Let us now use (35) to derive some more concrete Ortiz canonical polynomials. Taking into account that $(0,0) \in S$, it follows that $Q_{00}(x,y)$ is omitted, as being undefined; thus, $Q_{20}(x,y) = (1/2)x^2$. Similarly, we have $Q_{30}(x,y) = (1/6)x^3$. Since $Q_{01}(x,y) = (1/2)y^2$, as being previously defined by (34), we have $Q_{21}(x,y) = (1/2)\{x^2 y - y^2\}$. Let us now determine the form of the remaining residual polynomials. If $n = 2$, $m = 0$, then the first term of the right-hand side of (31) has index $(0,0) \in S$ and the second term is zero (the third term is not an element of neither S nor $I(2,0)$ and therefore, is not involved in (28)). Therefore, (28) gives $R_{20}(x,y) = 1$. Similarly, we have $R_{30}(x,y) = x$. Now, if $n \geq 4$, then the first term of the right-hand side of (31) has index $(n-2, 0) \in I(n,0) \setminus S$. Thus, (28) gives $R_{n0}(x,y) = -R_{n-2\,0}(x,y)$. Similarly, with the aid of (34), we have $R_{21}(x,y) = -R_{01}(x,y) = 0$, $R_{31}(x,y) = -R_{11}(x,y) = 0$, and $R_{n1}(x,y) = R_{n-2\,1}(x,y) = 0$ for $n \geq 4$. The above results are summarised in the formulae

$$
\begin{aligned}
R_{2n\,0}(x,y) &= (-1)^{n-1}, && \text{if } n \geq 1, \\
R_{2n+1\,0}(x,y) &= (-1)^{n-1} x, && \text{if } n \geq 1, \\
R_{n1}(x,y) &= 0, && \text{if } n \in \mathbf{N}_0.
\end{aligned} \tag{36}
$$

[10]Equivalently, we can write $Q_{nm}(x,y) = (1/m(m+1)) x^n y^{m+1}$; $R_{nm}(x,y) = 0$, for $n \leq 1$ and $m \geq 1$.

The remaining residual polynomials are defined by (28) as follows:

$$R_{nm}(x,y) = \frac{1}{n(n-1)}\left\{-m(m-1)R_{n\,m-1}(x,y) - n(n-1)R_{n-2\,m}(x,y)\right\}, \tag{37}$$

for $n \geq 2$ and $m \geq 2$. All previously defined residual polynomials, in the right-hand side of (37), are determined by either (34) or the third relation of (36). It follows that they must be zero, namely, $R_{nm}(x,y) = 0$ for all $n \geq 2$ and $m \geq 2$. Let us consider the differential problem

$$Dt(x,y) = p_{[nm]}(x,y), \tag{38}$$

where $p_{[nm]}(x,y)$ is a rectangular polynomial of the form $\sum_{i=0}^{n}\sum_{j=0}^{m}\alpha_{ij}x^i y^j$ with $\alpha_{nm} \neq 0$. Taking into account that the nonzero residual polynomials are those of index $(i,0)$ for $i \geq 2$, it follows from (19) that $p_{[nm]}(x,y)$ belongs to $\mathrm{im}(D)$ if and only if $\alpha_{00} + \alpha_{10}x = \sum_{i=2}^{n}\alpha_{i0}R_{i0}(x,y)$. Using the explicit form of $R_{i0}(x,y)$ in (36), the foregoing equation is equivalent to the following relations between the coefficients of $p_{[nm]}(x,y)$:

$$\alpha_{00} = \sum_{i=1}^{[n/2]}(-1)^{i-1}\alpha_{2i\,0}, \qquad \alpha_{10} = \sum_{i=1}^{[(n-1)/2]}(-1)^{i-1}\alpha_{2i+1\,0}, \tag{39}$$

where $[m]$ stands for the integer part of a rational number m. Given that the relations in (39) are satisfied, the bivariate polynomial solution of (38) takes the form

$$t(x,y) = \sum_{\substack{i=0\\(i,j)}}^{n}\sum_{\substack{j=0\\\notin S}}^{m}\alpha_{ij}Q_{ij}(x,y) + c_0 + c_1x + c_2y + c_3xy, \tag{40}$$

where c_i are free scalars, which could be adjusted by four supplementary conditions of the differential problem.

EXAMPLE 4. Let the differential operator be $D = (x^2+xy+y^2)\frac{\partial^2}{\partial x\,\partial y}$. The generating polynomials are given by

$$D\left(x^n y^m\right) = nmx^{n+1}y^{m-1} + nmx^n y^m + nmx^{n-1}y^{m+1}. \tag{41}$$

From (41), it follows that the indexing set W is the union of the sets $\{(n,0),\ n \in \mathbf{N}_0\}$ and $\{(0,m),\ m \in \mathbf{N}_0\}$, and the set complement J of W is $J = \{(n,m),\ n \geq 1,\ m \geq 1\}$. Let us now choose the ordering $\prec_{C1}$ of the indexing set I of $\mathbf{b}$. It follows from (41) that $\mathrm{maxsupp}(D(x^n y^m)) = (n-1, m+1)$ for $n \geq 1$ and $m \geq 1$ and so $\sigma(n,m) = (n-1, m+1)$. Formally, $\sigma : J \mapsto I$ is an injective mapping. Thus, $\mathrm{Ker}(D) = \mathrm{span}\{1, x^n, y^n,\ n \geq 1\}$. Also, the set of nonaccessible indices $S = I \setminus \sigma(J)$ is the union of the disjoint sets $\{(n,0),\ n \in \mathbf{N}_0\}$ and $\{(n,1),\ n \in \mathbf{N}_0\}$. Thus, $\mathcal{R}_S = \mathrm{span}\{1, x^n, x^n y,\ n \geq 1\}$. As $\mathrm{Ker}(D)$ and $\mathcal{R}_S$ are of infinite dimensions, the operator is of infinite Kernel index and deficiency. The inverse of σ generates the domain basis $\chi^\dagger = (1, x, x^2, \ldots, y, y^2, \ldots, xy, x^2y, xy^2, x^3y, x^2y^2, xy^3, \ldots)$. The matrix $[D]$, relative to $\chi^\dagger$, $\mathbf{b}$, is given by

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & \cdots \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3\cdots \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & \cdots
\end{pmatrix}.$$

The foregoing matrix is in pre-RE form, relative to Definition 2. If $m = 1$ and $n \geq 1$, then the first and the second terms of the right-hand side of (41) have indices in S, and so the corresponding Ortiz canonical and residual polynomials are $Q_{n-1\,2}(x,y) = (1/n)x^n y$ and $R_{n-1\,2}(x,y) = x^{n+1} + x^n y$. If $m = 2$ and $n \geq 1$, then only the first term of the right-hand side of (41) has index in S, and so the corresponding Ortiz canonical and residual polynomials are $Q_{n-1\,3}(x,y) = (1/2n)\{x^n y^2 - 2nQ_{n2}(x,y)\}$, and $R_{n-1\,3}(x,y) = x^{n+1}y - R_{n2}(x,y)$. For example, the residual polynomial of index $(0,3)$ is $R_{03}(x,y) = x^2y - R_{12}(x,y) = x^2y - (x^3 + x^2y) = -x^3$. If $m \geq 3$ and $n \geq 1$, then the remaining Ortiz canonical and residual polynomials are

$$\begin{aligned} Q_{n-1\,m+1}(x,y) &= \frac{1}{nm}\left\{x^n y^m - nmQ_{nm}(x,y) - nmQ_{n+1\,m-1}(x,y)\right\}, \\ R_{n-1\,m+1}(x,y) &= -R_{nm}(x,y) - R_{n+1\,m-1}(x,y). \end{aligned} \tag{42}$$

Let us consider the differential problem

$$Dt(x,y) = p_{\langle 03\rangle}(x,y), \tag{43}$$

where $p_{\langle 03\rangle}(x,y)$ is a triangular polynomial of index $(0,3)$, namely,

$$p_{\langle 03\rangle}(x,y) = \alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3.$$

It follows from (19) that $p_{\langle 03\rangle}(x,y)$ belongs to $\mathrm{im}(D)$ if and only if

$$\alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{30}x^3 + \alpha_{21}x^2y = \alpha_{02}R_{02}(x,y) + \alpha_{12}R_{12}(x,y) + \alpha_{03}R_{03}(x,y).$$

Substituting, in the above equation, the explicit form of the residual polynomials, we have

$$\alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{30}x^3 + \alpha_{21}x^2y = \alpha_{02}x^2 + \alpha_{02}xy + (\alpha_{12} - \alpha_{03})x^3 + \alpha_{12}x^2y,$$

and therefore, the above polynomial identity results in the following parametric relations:

$$\begin{aligned} \alpha_{00} &= \alpha_{10} = \alpha_{01} = 0, \\ \alpha_{20} &= \alpha_{11} = \alpha_{02} = c, \\ \alpha_{21} &= \alpha_{12} = d, \\ \alpha_{03} &= h, \\ \alpha_{30} &= d - h. \end{aligned}$$

Consequently, $p_{\langle 03\rangle}(x,y) \in \mathrm{im}(D)$ if and only if

$$p_{\langle 03\rangle}(x,y) = cx^2 + cxy + cy^2 + dxy^2 + dx^2y + hy^3 + (d-h)y^3,$$

for any scalars c, d, h.

Thus, the exact bivariate polynomial solution of (43) is given by

$$t(x,y) = \alpha_0 + \sum_{k\in\mathbf{N}} \alpha_k x^k + \sum_{k\in\mathbf{N}} \beta_k y^k + cQ_{02}(x,y) + dQ_{22}(x,y) + hQ_{03}(x,y),$$

where α_k, β_k are free coefficients and $\mathbf{N} = \{1,2,3,\dots\}$.

REFERENCES

1. E.L. Ortiz, On the generation of the canonical polynomials associated with certain linear differential operators, Research Report, pp. 1–21, Imperial College, London, (1964).
2. E.L. Ortiz, The Tau method, *SIAM J. Numer. Anal.* **6**, 480–491, (1969).
3. P. Llorente and E.L. Ortiz, Sur quelques aspécts algébriques d' une méthode de M. Lanczos, *Math. Notae* **21**, 17–23, (1968).
4. E.L. Ortiz and H. Samara, An operational approach to the Tau method for the numerical solution of nonlinear differential equations, *Computing* **27**, 15–25, (1981).
5. C. Lanczos, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, NJ, (1956).
6. E.L. Ortiz, Canonical polynomials in the Lanczos Tau method, In *Studies in Numerical Analysis*, pp. 73–93, Academic Press, London, (1974).
7. E.L. Ortiz and H. Samara, Numerical solution of partial differential equations with variable coefficients with an operational approach to the Tau method, *Computers Math. Applic.* **10** (1), 5–13, (1984).
8. A.G. Paraskevopoulos, The minimal residual space in the vector formulation of the Tau method (to appear).
9. A. Abadi, M. Hosseini and E.L. Ortiz, The algebraic kernel method for the numerical solution of partial differential equations, *Numer. Funct. Anal. and Optimiz.* **12**, 339–360, (1991).
10. A.G. Paraskevopoulos, Existence and uniqueness of reduced row echelon matrices associated with abstract linear mappings (to appear).
11. N. Bourbaki, *Elements of Mathematics, Theory of Sets*, Adiwes International Series in Mathematics, Herman, (1968).
12. N. Jacobson, *Lectures in Abstract Algebra, Volume II—Linear Algebra*, D. Van Nostrand, (1953).
13. N. Bourbaki, *Elements of Mathematics, Algebra I*, Adiwes International Series in Mathematics, Herman, (1974).
14. A.G. Paraskevopoulos, On the decomposition of a matrix differential operator with polynomial coefficients and the vector polynomial solutions of ODEs, *Proceedings of the Cornelius Lanczos International Centenary Conference, SIAM. J.*, 347–352, (1994).